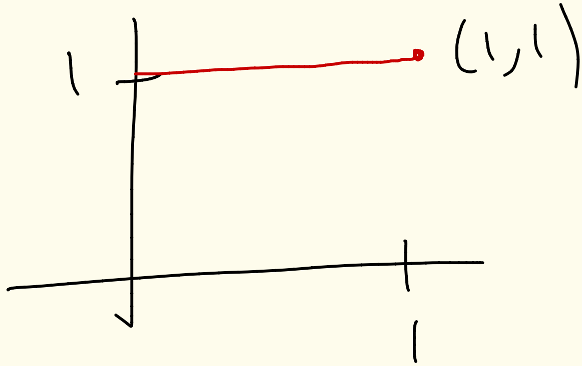
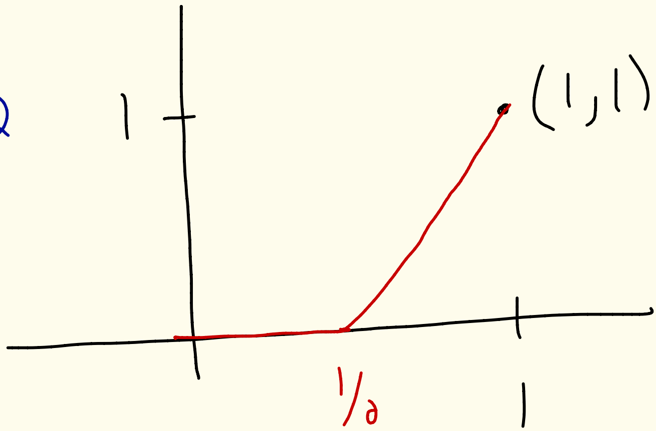
Difference Between Pointwise + Uniform Convergence

Pointwise: speed of convergence depends on location of the domain

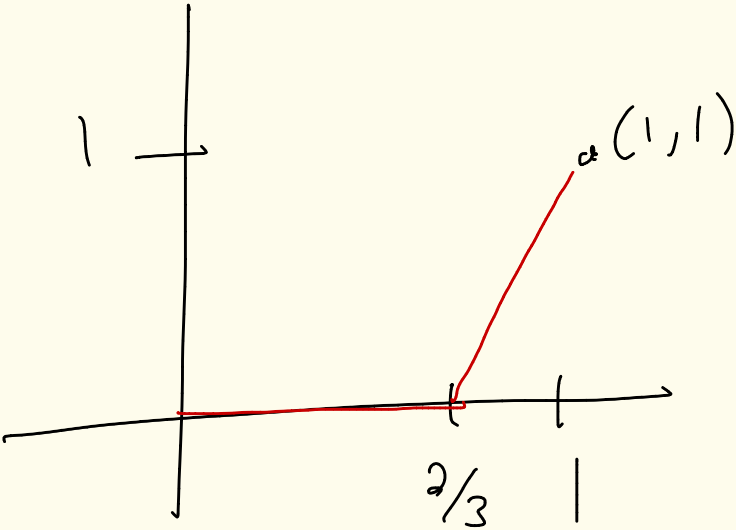
f_1



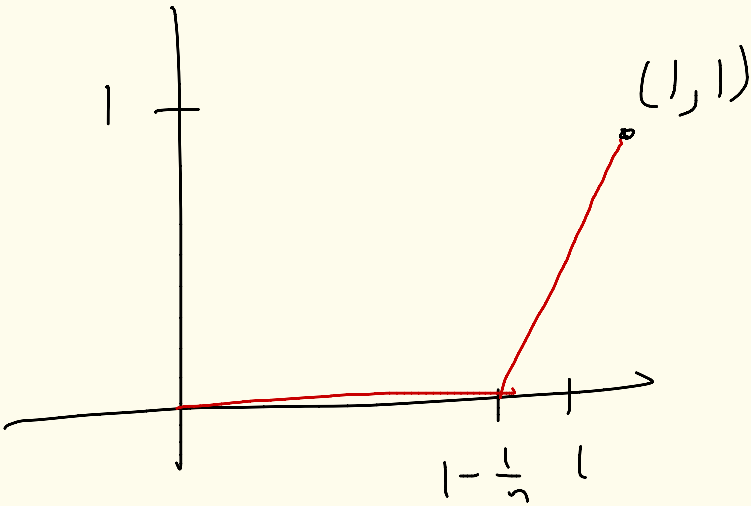
f_2



f_3



f_n



On $[0,1)$

$f_n \to 0$ pointwise

but not uniformly

since if $x \in [0,1)$,

choose a minimal value

of n with

$$1 - \frac{1}{n} > x$$

For all $m \geq n$,

$f_n(x) = 0$, but if

$m < n$, $f_m(x) \neq 0$

and maybe close to 1!

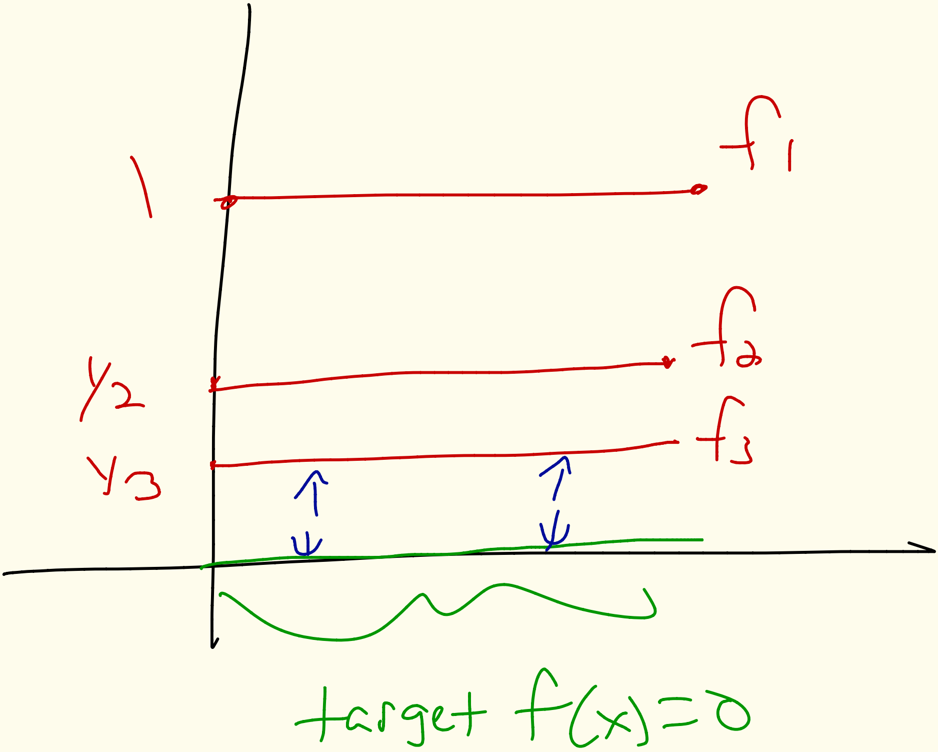
Uniform Convergence

Let $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = \frac{1}{n}.$$

$f_n \to 0$ pointwise

but also uniformly



graphs get close to zero on the same order everywhere on the interval, independent of choice of x.

Lemma: (substitution)

Let $g : [a,b] \to \mathbb{R}$ be differentiable and let $f : g([a,b]) \to \mathbb{R}$ be continuous. Then if g' is integrable,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(x)\,dx$$

Proof: Let h be any antiderivative of f on the interval $[g(a), g(b)]$, which exists by the Fundamental Theorem of Calculus. Then by the chain rule, $\forall\, x \in [a,b]$,

$$\begin{aligned}(h(g(x)))' &= h'(g(x))\, g'(x) \\ &= f(g(x))\, g'(x).\end{aligned}$$

Then $h(g(x))$ is an antiderivative for $f(g(x)) \cdot g'(x)$ on $[a,b]$, so by the Fundamental Theorem,

$$\int_a^b f(g(x)) \cdot g'(x)\,dx$$

$$= h(g(b)) - h(g(a))$$

$$= \int_{g(a)}^{g(b)} f(x)\,dx. \quad \square$$

Lemma: (polynomial inequality)

Suppose $0 \leq x \leq 1$.

Then $(1-x^2)^n \geq 1-nx^2$

$\forall\, n \in \mathbb{N}$.

proof: $n=1$, the statement is trivial. Suppose $n>1$.

Let $f_n(x) = (1-x^2)^n - (1-nx^2)$.

Show $f_n(x) \geq 0 \;\forall\, n>1$.

We see that

$$f_n(0) = 0 \quad \forall\, n > 1.$$

Consider

$$f_n'(x) = n(1-x^2)^{n-1} \cdot 2x + 2nx$$

$$> 0 \quad \forall\, x \in (0,1]$$

Since $1-x^2 \geq 0$ for all such x.

This shows f_n is increasing

$\forall\, n \in \mathbb{N}$ on $[0,1]$

Then

$f_n \geq 0 \quad \forall\, n \in \mathbb{N}$ on

$[0,1]$

$\Rightarrow (1-x^2)^n \geq 1-nx^2.$

$\square$

Theorem: (Weierstrass Approximation)

Let f be a continuous, real-valued function on $[a,b]$.

Then $\exists$ a sequence of polynomials $(p_n)_{n=1}^{\infty}$ such that $p_n \to f$ uniformly on $[a,b]$.

Proof: Assume that

$a = 0$, $b = 1$, and $f(0) = f(1) = 0$.

We'll show the theorem holds in this case and then address why this is sufficient.

Extend f to $\mathbb{R}$ by defining $f(x) = 0 \ \forall x \notin [0,1]$.

Since $f(0) = f(1) = 0$, the resulting function is continuous.

Define $Q_n(t) = c_n(1-t^2)^n$

where $c_n = \dfrac{1}{\int_{-1}^{1} (1-t^2)^n \, dt}$.

(Note: $\int_{-1}^{1} Q_n(t)\,dt = 1$)

Define

$$P_n(x) = \int_{-1}^{1} f(x+t)\,Q_n(t)\,dt$$

Claims:

1) P_n is a polynomial $\forall n \in \mathbb{N}$

2) $P_n \to f$ uniformly on $[0,1]$.

1)

$$P_n(x) = \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt$$

$$= \int_{-x}^{1-x} f(x+t)\, Q_n(t)\, dt$$

Since $f(x+(1-x)) = f(1)$

$= 0$

$= f(0)$

$= f(x-x)$

and $f = 0$ for all $y \notin [0,1]$.

$$\int_{-x}^{1-x} f(x+t)\, Q_n(t)\, dt$$

$$u(t) = x+t$$

$$u'(t) = 1$$

$u(-x) = 0$ and

$u(1-x) = 1$. Then by

substitution,

$$P_n(x) = \int_0^1 f(u)\, Q_n(u-x)\, du$$

Now

$$Q_n(x) = (1-x^2)^n \text{, so}$$

$$Q_n(u-x) = (1-(u-x)^2)^n$$

$$= \sum_{k=0}^{n} (-1)^k (u-x)^{2k} \binom{n}{k}$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{m=0}^{2k} (-1)^m \binom{2k}{m} u^m x^{2k-m}$$

$$= \sum_{k=0}^{n} \sum_{m=0}^{2k} (-1)^{k+m} \binom{n}{k} \binom{2k}{m} u^m x^{2k-m}$$

Then

$$P_n(x) = \int_{-1}^{1} f(u)\, Q_n(u-x)\, du$$

$$= \int_{-1}^{1} f(u) \sum_{k=0}^{n} \sum_{m=0}^{2k} (-1)^{k+m} \binom{n}{k} \binom{2k}{m} u^m x^{2k-m}\, du$$

$$= \sum_{k=0}^{n} \sum_{m=0}^{2k} \overbrace{\left(\int_{-1}^{1} f(u) (-1)^{k+m} \binom{n}{k} \binom{2k}{m} u^m\, du \right)}^{\text{some number}} \cdot x^{2k-m}$$

is a polynomial!